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Published in:
IFAC-PapersOnLine

DOI:
[10.1016/j.ifacol.2017.08.920](https://doi.org/10.1016/j.ifacol.2017.08.920)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Final author's version (accepted by publisher, after peer review)

Publication date:
2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Kawano, Y., & Scherpen, J. M. A. (2017). Empirical Differential Balancing for Nonlinear Systems. In D. Dochain, D. Henrion, & D. Peaucelle (Eds.), IFAC-PapersOnLine (pp. 6326-6331). (IFAC - Papers OnLine; Vol. 50, No. 1). IFAC. <https://doi.org/10.1016/j.ifacol.2017.08.920>

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Empirical Differential Balancing for Nonlinear Systems

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Abstract: In this paper, we consider empirical balancing of a nonlinear system by using its prolonged system, which consists of the original nonlinear system and its variational system. For the prolonged system, we define differential reachability and observability Gramians, which are matrix valued functions of the state trajectory (i.e. the initial state and input trajectory) of the original system. The main result of this paper is showing that for a fixed state trajectory, it is possible to compute the values of these Gramians by using impulse and initial state responses of the variational system. By using the obtained values of the Gramians, balanced truncation is doable along the fixed state trajectory without solving nonlinear partial differential equations. An example demonstrates our proposed method to compute a reduced order model along a limit cycle of a coupled van der Pol oscillator.

Keywords: Nonlinear systems, prolongation, empirical Gramians, balancing

1. INTRODUCTION

Model order reduction problems have been widely studied because reduced order models are useful for analysis and controller design. In both linear and nonlinear control theory, balanced truncation (Antoulas (2005); Zhou et al. (1996); Scherpen (1993); Fujimoto and Scherpen (2005); Besselink et al. (2014); Kawano and Scherpen (2015, 2017)) is known as a traditional model reduction method. Besides balancing, also moment matching (Antoulas (2005)) is a well known tool for model reduction. For nonlinear control systems this method has only been recently developed, see Astolfi (2010); Ionescu and Astolfi (2016). However, there still remains the problem that solutions to nonlinear PDE (partial differential equations) are required for both balanced truncation and moment matching. In the field of engineering, especially fluid mechanics, POD (proper orthogonal decomposition) (Jolliffe (2002)) is often used for model reduction of nonlinear dynamical systems. POD is based on data, i.e., POD does not require a solution to a PDE. However, theoretical analysis of POD is not well developed, and this method is proposed only for noncontrolled systems.

For linear systems, POD and balancing is connected based on the fact that the controllability and observability Gramians can be computed by using impulse and initial responses, respectively. That is, balanced truncation of linear systems can be performed by using empirical data. This empirical method is applicable for nonlinear systems as demonstrated for mechanical links in Lall et al. (2002). However, the relationship between impulse and initial responses of nonlinear systems and nonlinear balancing is not studied. On the other hand, recently, a connection between POD and nonlinear controllability functions is

established by Kashima (2016), but the observability function is not studied.

In this paper, we propose an empirical balancing method for nonlinear systems with constant input vector fields and output functions. To this end, we utilize the prolonged system, which consists of the original nonlinear system and its variational system. First, we define two Gramians for the prolonged system, which we call differential reachability and observability Gramians and which depend on the state trajectory of the original system. In general, it is not easy to obtain these differential Gramians as nonlinear functions of the state trajectory. However, as will be shown in this paper, for each fixed state trajectory, it is possible to compute the values of these Gramians by using impulse and initial state responses of the variational system along this state trajectory. Then, the obtained trajectory-wise Gramians are constant matrices. By using these Gramians, we can compute balanced coordinates and thus a reduced order model in a similar manner as the linear case. A reduced order model may not give a good approximation for an arbitrary state trajectory but it does along the fixed state trajectory. In summary, by using our method, model reduction of a nonlinear control system can be performed using empirical data.

Similar ideas for model reduction of nonlinear systems are found in flow balancing (Verriest and Gray (2000, 2004)) and another balancing in Kawano and Scherpen (2017). For flow balancing, the input is fixed for any initial state. However, in order to compute the Gramians, we need to solve nonlinear PDEs. Kawano and Scherpen (2017) do not give a concept of Gramian, and the balanced coordinate is defined by using solutions to PDEs. Thus, neither flow balancing nor balancing in Kawano and Scherpen (2017) is an empirical balancing method.

The remainder of this paper is organized as follows. In Section 2, we give a review of linear empirical balancing. In Section 3, we define differential reachability and observability Gramians. By using these Gramians, we define a differentially balanced realization along a trajectory of system. Next, we show that the values of the differential reachability and observability Gramians can be computed by using impulse and initial state responses of the variational system. An example for a coupled van der Pol oscillator illustrates our method. Finally in Section 5, we conclude the paper.

2. REVIEW OF LINEAR EMPIRICAL BALANCED TRUNCATION

Our aim is to extend linear empirical balancing to nonlinear systems. To be self contained, we provide a brief summary of the linear results.

Consider the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. For this system, the controllability and observability Gramians are respectively defined as

$$\begin{aligned} G_c &= \int_0^\infty e^{At} B B^T e^{A^T t} dt, \\ G_o &= \int_0^\infty e^{A^T t} C^T C e^{At} dt. \end{aligned}$$

Suppose that the system is asymptotically stable at the origin, controllable, and observable. Then, these Gramians are unique positive definite solutions to the Lyapunov equations. For positive definite controllability and observability Gramians, there always exists a so-called balanced coordinate $z = Tx$ such that

$$T^{-1} G_c T^{-T} = T^T G_o T = \text{diag}\{\sigma_1, \dots, \sigma_n\}, \quad \sigma_i \geq \sigma_{i+1},$$

where $T^{-1} G_c T^{-T}$ and $T^T G_o T$ are the controllability and observability Gramians in the z -coordinates, respectively. See e.g. Antoulas (2005).

As explained above, the balanced coordinates can be obtained by computing the controllability and observability Gramians, i.e., solving the Lyapunov equations. In fact, from their definitions, these Gramians can also be computed by using the impulse and initial state responses of the system. First, the (zero initial state) impulse response of the linear system is

$$x_{\text{Imp}}(t) = e^{At} B.$$

Then, the controllability Gramian can be rearranged as

$$G_c = \int_0^\infty x_{\text{Imp}}(t) x_{\text{Imp}}^T(t) dt,$$

which implies that the controllability Gramian can be constructed by using the impulse response.

Next, let $\eta^i(t)$ be the (zero input) initial state response with the initial state e_i , which is a standard basis, i.e.,

whose i th element is 1, and the other elements are the zeros. Then, $\eta^i(t)$ can be described as

$$\eta^i(t) = C e^{At} e_i.$$

By using this $y_i(t)$, the observability Gramian can be rearranged as

$$G_o = \int_0^\infty [\eta^1(t) \cdots \eta^n(t)]^T [\eta^1(t) \cdots \eta^n(t)] dt.$$

That is, the observability Gramian can be constructed by using the initial state responses. From these facts, the balanced coordinates can be computed from empirical data without solving the Lyapunov equations.

3. EMPIRICAL DIFFERENTIAL BALANCING

3.1 Differential Reachability and Observability Gramians

In this paper, we present a nonlinear empirical balancing method by using the prolonged system, which consists of the original nonlinear system and its variational system. First, we define a nonlinear differentially balanced realization for the prolonged system, then extend the linear empirical method to this differential balancing.

Consider the following nonlinear system with the constant input vector field and output function

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$; $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is sufficiently smooth, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$. Let $\varphi_{t-t_0}(x_0, u)$ denote the state trajectory $x(t)$ of the system Σ starting from $x(t_0) = x_0 \in \mathbb{R}^n$ for each choice of $u \in L_2^m[t_0, \infty)$.

In our method, we use the prolonged system of the system Σ , which consists of the original system Σ and its variational system $d\Sigma$ along $x(t)$,

$$d\Sigma : \begin{cases} \delta \dot{x}(t) := \frac{d\delta x(t)}{dt} = \frac{\partial f(x(t))}{\partial x} \delta x(t) + B \delta u(t), \\ \delta y(t) = C \delta x(t), \end{cases}$$

where $\delta x(t) \in \mathbb{R}^n$, $\delta u(t) \in \mathbb{R}^m$ and $\delta y(t) \in \mathbb{R}^p$. In the time interval $[t_0, t_f]$ such that $x(t)$ exists and is smooth, $\delta x(t)$ exists for any bounded input $\delta u(t)$ because the variational system $d\Sigma$ is a linear time varying system along $x(t)$.

Remark 1. Let $x(t, \varepsilon)$, $u(t, \varepsilon)$, and $y(t, \varepsilon)$, $t \in [t_0, t_f]$ be a family of input-state-output trajectories of a system Σ parametrized in $\varepsilon \in (-\delta, \delta)$, with $x(t, 0) = x(t)$, $u(t, 0) = u(t)$, and $y(t, 0) = y(t)$. Then, the infinitesimal variations $\delta x(t) = \partial x(t, \varepsilon) / \varepsilon$, $\delta u(t) = \partial u(t, \varepsilon) / \varepsilon$, and $\delta y(t) = \partial y(t, \varepsilon) / \varepsilon$ satisfy the system equation for $d\Sigma$. Therefore, the system $d\Sigma$ is called the variational system and is used for variational analysis of the trajectory of the original system Σ . \triangleleft

To extend the linear empirical method to the variational system along a given trajectory $x(t)$ of the original system Σ , first we consider to compute the solution $\delta x(t)$ of $d\Sigma$. For solution $x(t) = \varphi_{t-\tau}(x_\tau, u)$ of the original system Σ starting from $x(\tau) = x_\tau \in \mathbb{R}^n$ with the input $u \in L_2^m[\tau, t]$, we have

$$\frac{d}{dt} \frac{\partial \varphi_{t-\tau}(x_\tau, u)}{\partial x_\tau} = \frac{\partial f(x(t))}{\partial x} \frac{\partial \varphi_{t-\tau}(x_\tau, u)}{\partial x_\tau}. \quad (1)$$

That is, $\partial \varphi_{t-\tau}(x_\tau, u)/\partial x_\tau$ is the transition matrix of $\partial f(x(t))/\partial x$. We also call it the transition matrix of the variational system $d\Sigma$ along $x(t) = \varphi_{t-\tau}(x_\tau, u)$, since the variational system is a linear time-varying system along trajectory $x(t)$. By using this transition matrix, the solution $\delta x(t)$ of the variational system $d\Sigma$ starting from $\delta x(t_0) = \delta x_0$ with input $\delta u(t)$ along $x(t)$ can be described as follows with the well known theory.

$$\delta x(t) = \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} \delta x_0 + \int_{t_0}^t \frac{\partial \varphi_{t-\tau}(x(\tau), u)}{\partial x} B \delta u(\tau) d\tau. \quad (2)$$

By using the transition matrix of the variational system $d\Sigma$, we define the differential Gramians. Then, by using these Gramians, we define a balanced realization.

Definition 2. The differential reachability Gramian is defined as

$$G_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} B B^T \frac{\partial^T \varphi_{t-t_0}(x_0, u)}{\partial x} dt \quad (3)$$

for $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$.

Definition 3. The differential observability Gramian is defined as

$$G_{\mathcal{O}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \frac{\partial^T \varphi_{t-t_0}(x_0, u)}{\partial x} C^T C \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} dt \quad (4)$$

for $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$.

The existence of these differential Gramians is not guaranteed for arbitrary t_0, t_f and $x(t)$. However, from their definitions, if $x(t)$ exists and is smooth in time interval $[t_0, t_f]$, these differential Gramians exist.

Similar Gramians are found in flow balancing (Verriest and Gray (2000, 2004)). For flow balancing, the reachability and observability Gramians are defined on different time intervals, and the input is fixed for any initial state. Thus, the Gramians for flow balancing are defined as functions of the initial states. In contrast, our differential Gramians also depend on the input trajectory in addition to the initial state.

Remark 4. For a linear time-invariant system, the differential reachability and observability Gramians reduce to

$$G_c = \int_{t_0}^{t_f} e^{A(t-t_0)} B B^T e^{A^T(t-t_0)} dt, \\ G_o = \int_{t_0}^{t_f} e^{A^T(t-t_0)} C^T C e^{A(t-t_0)} dt.$$

These are controllability and observability Gramians on a finite time interval (Antoulas (2005)). These Gramians also exist for systems that are not necessarily asymptotically stable. \triangleleft

Remark 5. By substituting $t = t_f + t_0 - \tau$, the differential reachability Gramian can be rearranged as

$$G_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \frac{\partial \varphi_{t_f-\tau}(x_f, \mathcal{F}_-(u))}{\partial x} B B^T \frac{\partial^T \varphi_{t_f-\tau}(x_f, \mathcal{F}_-(u))}{\partial x} d\tau,$$

where $\varphi_{t_f-\tau}(x_f, \mathcal{F}_-(u))$ is the backward trajectory of system Σ starting from $x(t_f) = x_f$ with the input $\mathcal{F}_-(u) = u(t_f + t_0 - \tau) \in L_2^m[t_0, t_f]$. This is an extension of the reachability Gramian for linear time-varying systems in Verriest and Kailath (1983) to nonlinear prolonged systems. The differential observability Gramian is an extended concept of the observability Gramian for linear time-varying systems as well. \triangleleft

3.2 Differentially Balanced Realization along a Fixed State Trajectory

The differential reachability and observability Gramians are functions of x_0 and u . If we can obtain these Gramians as functions of x_0 and u , it is possible to construct a nonlinear balanced coordinate transformation. However, this is a difficult task. For instance, for flow balancing, we need to solve nonlinear partial differential equations. In this paper, we consider to find a linear balanced coordinate transformation along a fixed state trajectory $\varphi_{t-t_0}(x_0, u)$. The obtained reduced order model may not approximate the original model very well for any state trajectory but it does at least locally around $\varphi_{t-t_0}(x_0, u)$.

In this paper, we consider the following balanced realization of the differential reachability and observability Gramians.

Definition 6. Let the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$ and differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$ at fixed $\varphi_{t-t_0}(x_0, u)$ be positive definite. A realization of the system Σ is said to be a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ if there exists a constant diagonal matrix

$$\Lambda = \text{diag}\{\sigma_1, \dots, \sigma_n\},$$

where $\sigma_1 \geq \dots \geq \sigma_n > 0$ holds, and $G_{\mathcal{R}}(t_0, t_f, x, u) = \Lambda$ and $G_{\mathcal{O}}(t_0, t_f, x, u) = \Lambda$.

In a similar manner as for the linear time-invariant case (Antoulas (2005)), it is possible to show the existence of a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ if positive definite differential reachability and observability Gramians exist.

Theorem 7. Let the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$ and differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$ at fixed $\varphi_{t-t_0}(x_0, u)$ be positive definite. Then, there exists a constant matrix T which achieves

$$T G_{\mathcal{R}}(t_0, t_f, x_0, u) T^T = T^{-T} G_{\mathcal{O}}(t_0, t_f, x_0, u) T^{-1} \\ = \Lambda := \text{diag}\{\sigma_1, \dots, \sigma_n\},$$

where $\sigma_1 \geq \dots \geq \sigma_n > 0$. Then a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ is obtained after a coordinate transformation $z = Tx$. \triangleleft

If we can compute the differential reachability and observability Gramians along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$, then a differentially balanced realization along this trajectory is computed by using a linear coordinate transformation. Clearly, this linear coordinate transformation depends on the trajectory.

The differentially balanced realization is defined for positive definite differential reachability and observability Gramians. In a specific case when $u \equiv 0$, local accessibility and observability of the original nonlinear system are sufficient conditions for positive definiteness; See Nijmeijer and van der Schaft (1990) for the definitions of local strong accessibility and observability.

Proposition 8. Suppose that the strong accessibility distribution (Nijmeijer and van der Schaft (1990)) of a system Σ has a constant dimension. If a system Σ is locally strongly accessible, and solution $x(t)$ of Σ exists for any x_0 and $u \equiv 0$ in a time interval $[t_0, t_f]$, then the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ is positive definite for any x_0 and $u \equiv 0$.

Proposition 9. Let $u \equiv 0$ and $\delta u \equiv 0$. Suppose that the observability codistribution (Nijmeijer and van der Schaft (1990)) of this system Σ has a constant dimension. If this system Σ is locally observable, and solution $x(t)$ of Σ exists for any x_0 in a time interval $[t_0, t_f]$, then the differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ is positive definite for any x_0 and $u \equiv 0$.

3.3 Empirical Methods for Differential Gramians

This section is dedicated to present trajectory-wise computational methods of the differential Gramians. First, we show that the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$ can be computed by using an impulse response of $d\Sigma$. Define

$$\delta x_{\text{Imp}}(t) := \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} B.$$

Then, from the definition (3) of the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u)$, we have

$$G_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \delta x_{\text{Imp}}(t) \delta x_{\text{Imp}}^T(t) dt. \quad (5)$$

By using Dirac's delta function $\delta(\cdot)$, $\delta x_{\text{Im}}(t)$ can be represented as

$$\delta x_{\text{Imp}}(t) = \int_{t_0}^t \frac{\partial \varphi_{t-\tau}(x(\tau), u)}{\partial x} B \delta(\tau - t_0) d\tau. \quad (6)$$

From (2), this is the impulse response of $d\Sigma$ starting from the initial state $\delta x_{\text{Im}}(t_0) = 0$ along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$. Therefore, for each $x(t_0) = x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$, the constant matrix $G_{\mathcal{R}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$, a trajectory-wise differential reachability Gramian is obtained by using the impulse response of $d\Sigma$.

Next, we show that differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ can be computed by using initial state responses. Denote

$$\delta \eta^i(t) = C \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} e_i, \quad i = 1, \dots, n, \quad (7)$$

where e_i is the standard basis. This is the initial state response of $d\Sigma$ starting from the initial state $\delta x(t_0) = e_i$ with input $\delta u = 0$ along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$. From the definition (4) of differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u)$, we have

$$G_{\mathcal{O}}(t_0, t_f, x_0, u) := \int_{t_0}^{t_f} [\delta \eta^1(t) \cdots \delta \eta^n(t)]^T \times [\delta \eta^1(t) \cdots \delta \eta^n(t)] dt. \quad (8)$$

Therefore, the differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n \times n}$ can be computed trajectory-wise by using initial state responses of $d\Sigma$ starting from $\delta x(t) = e_i$ for each $x(t_0) = x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$.

4. EXAMPLE

In this example, we consider differentially balanced truncation of the following coupled van der Pol oscillator along a limit cycle.

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \mu(x_1^2 - 1)x_2 + a(x_3 - x_1) + b(x_4 - x_2) + u, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -x_3 - \mu(x_3^2 - 1)x_4 + a(x_1 - x_3) + b(x_2 - x_4) + u, \\ y_1 &= x_1, \\ y_2 &= x_2, \end{aligned}$$

where $a = 0.5$, $b = 0.2$, and $\mu = 0.5$. The variational system is

$$\begin{aligned} \delta \dot{x}_1 &= \delta x_2, \\ \delta \dot{x}_2 &= -(1 + 2\mu x_1 x_2) \delta x_1 - \mu(x_1^2 - 1) \delta x_2 \\ &\quad + a(\delta x_3 - \delta x_1) + b(\delta x_4 - \delta x_2) + \delta u, \\ \delta \dot{x}_3 &= \delta x_4, \\ \delta \dot{x}_4 &= -(1 + 2\mu x_3 x_4) \delta x_3 - \mu(x_3^2 - 1) \delta x_4, \\ &\quad + a(\delta x_1 - \delta x_3) + b(\delta x_2 - \delta x_4) + \delta u, \\ \delta y_1 &= \delta x_1, \\ \delta y_2 &= \delta x_2. \end{aligned}$$

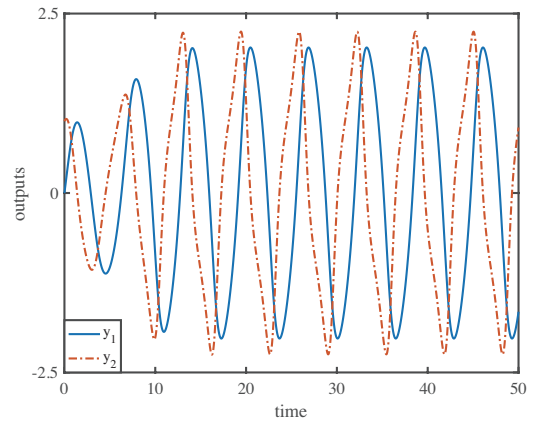


Fig. 1. Output trajectories

Figure 1 shows output trajectories y_1 and y_2 starting from $x_0 := [0 \ 1 \ 0 \ 0]^T$ with zero input. This coupled van der Pol oscillator has a limit cycle. We consider empirical differentially balanced truncation along this limit cycle. According to Fig. 1, the outputs converge to the limit cycle after $t = 20$. Therefore, we compute the empirical differential reachability and observability Gramians in the time interval $[20, 50]$, which are obtained as follows.

$$G_{\mathcal{R}}(20, 50, x_0, 0) = \begin{bmatrix} 10.3 & 0.00993 & 10.3 & 0.00968 \\ 0.00993 & 12.9 & 0.0102 & 12.9 \\ 10.3 & 0.0102 & 10.3 & 0.00994 \\ 0.00968 & 12.9 & 0.00994 & 12.9 \end{bmatrix},$$

$$G_{\mathcal{O}}(20, 50, x_0, 0) = 10^2 \times \begin{bmatrix} 197 & 5.29 & 194 & 3.62 \\ 5.29 & 4.83 & 8.46 & 4.72 \\ 194 & 8.46 & 199 & 7.06 \\ 3.62 & 4.72 & 7.06 & 10.3 \end{bmatrix}.$$

Then, the singular value matrix Λ in Theorem 7 is computed as

$$\Lambda = \begin{bmatrix} 8.13 \times 10^5 & 0 & 0 & 0 \\ 0 & 3.08 \times 10^4 & 0 & 0 \\ 0 & 0 & 4.89 \times 10^{-5} & 0 \\ 0 & 0 & 0 & 1.25 \times 10^{-5} \end{bmatrix}.$$

Since $\sigma_2 \gg \sigma_3$, to give an approximation of the limit cycle, we compute a two-dimensional reduced order model. The obtained reduced order model by our empirical method is

$$\begin{aligned} \dot{z}_{r,1} &= 1.18 \times 10^{-2} z_{r,1} + 1.02 z_{r,2} + 4.89 \times 10^{-4} z_{r,2}^2 z_{r,1} \\ &\quad - 7.53 \times 10^{-3} z_{r,1}^2 z_{r,2} - 3.13 \times 10^{-4} z_{r,1}^3 \\ &\quad - 7.93 \times 10^{-6} z_{r,2}^3 - 4.58 \times 10^{-2} u, \\ \dot{z}_{r,2} &= -0.979 z_{r,1} + 0.490 z_{r,2} + 1.61 \times 10^{-2} z_{r,2}^2 z_{r,1} \\ &\quad - 0.249 z_{r,1}^2 z_{r,2} - 1.03 \times 10^{-2} z_{r,1}^3 - 2.62 \times 10^{-4} z_{r,2}^3 \\ &\quad - 1.41 u, \\ y_{r,1} &= -0.707 z_{r,1} - 2.29 \times 10^{-2} z_{r,2}, \\ y_{r,2} &= -2.93 \times 10^{-2} z_{r,1} - 0.707 z_{r,2}. \end{aligned}$$

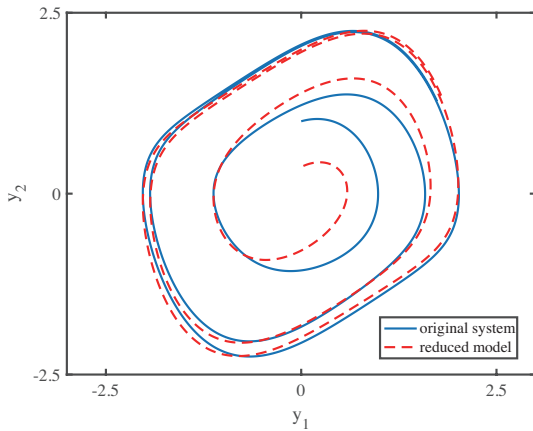


Fig. 2. Phase portraits of the original system and reduced order model

Figure 2 shows the phase portraits of the original system and reduced order model, where the initial state of the original system is $x(0) := [0 \ 1 \ 0 \ 0]^T$, and the initial state $z(0)$ of the reduced order model is chosen to coincide with $x(0)$, and the input is zero. According to Fig. 2, the reduced order model gives a good approximation of a limit cycle.

5. CONCLUSION

In this paper, we propose an empirical balancing method for nonlinear prolonged systems along a fixed trajectory. This method is based on the differential reachability and observability Gramians. These Gramians are the functions of initial state and input trajectory. Based on these Gramians, we defined a differentially balanced realization to give a reduced order model around the fixed trajectory. The main result of this paper is showing that this differentially balanced realization along a fixed state trajectory can be computed by using impulse and initial state responses of the variational system along the state trajectory. That is, we do not need to solve any nonlinear partial differential equation in contrast to conventional nonlinear balancing methods.

Future work includes constructing an empirical method without the variational system and establishing a connection between the proposed method and differential balancing presented in Kawano and Scherpen (2017). At present, our method requires the variational system, and its computation is not difficult. However, if systems become large-scale, we need to make an effort to compute the variational system. Therefore, an empirical method which is doable only by using the original system is more useful.

On the other hand, in Kawano and Scherpen (2017), a differential balancing is proposed, which is based on contraction theory, and thus uses the prolonged system. This differential balancing is based on two energy functions, the so called differential controllability and observability functions. At present, the relationship between these differential energy functions and differential Gramians defined in this paper is not clear. Clarifying this relationship is a future work.

ACKNOWLEDGMENT

This work of Y. Kawano was partly supported by JST CREST Grant Number JPMJCR15K2, Japan.

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